A NOTE ON AUTOMORPHISMS OF FREE NILPOTENT GROUPS

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Abstract. We exhibit normal subgroups of a free nilpotent group F of rank two and class three, which have isomorphic finite quotients but are not conjugate under any automorphism of F.

A remarkable fact about free profinite groups of finite rank is that any isomorphism between finite quotients of such a group F lifts to an automorphism of F. This is true, more generally, if F a free pro-$\mathcal{C}$-group of finite rank, where $\mathcal{C}$ is a family of finite groups closed under taking subgroups, homomorphic images and direct products, and containing nontrivial groups. A proof and the relevant definitions can be found in [FJ86, Proposition 15.31], but the essence of the argument goes back to Gaschütz [Gas55]. In preparation for a summer school on *"Zeta functions of groups"* held by Marcus du Sautoy and the author in June 2002 in Trento (Italy), du Sautoy suggested that this may remain true for (abstract) free nilpotent groups F, and asked the author, who was responsible for that part of the course, to provide a proof. If confirmed, this claim would have simplified the course by avoiding the need to set up the language of profinite groups.

Unfortunately, this claim already fails for F a free abelian group of rank one, that is, an infinite cyclic group: in this case F has exactly two automorphism, while its quotient of order n has $\varphi(n)$ automorphisms, and $\varphi(n) > 2$ for $n > 4$. A milder statement which would have been sufficient for our purposes would be that any two normal subgroups of F with finite isomorphic quotients are conjugate under some automorphism of F. This is also false, and one does not have to dig much deeper in order to find a counterexample. We first record an example suggested by the anonymous referee. It is based on a three-generated group of order p^6 and class two, which was studied in [DH75]. After that we present an example where F is two-generated and the quotients have order p^4. This order is easily seen to be minimal for such an example.

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Example. The groups G of odd order p^6 satisfying $G' = Z(G) = G^p$ were classified by Daues and Heineken in [DH75] in terms of dualities of a three-dimensional vector space over the field of p elements. In particular, the group G in their case (I) has a p-group as the full group of automorphisms. One can realize G as the quotient of the free nilpotent group $F = \langle x, y, z\rangle$ of rank three and class two modulo the normal subgroup M_r generated by $(F')^p$ and the three elements

$$x^{rp}[y,x], \quad y^{rp}[z,x], \quad z^{rp}[z,x]^{-1}[z,y],$$

where r is any integer prime to p. When $r = 1$ the relations associated with these three elements correspond to the matrix $D = (a_{ij}) = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{smallmatrix}\right)$ as described in [DH75, p. 219], with x, y, z in place of x_1, x_2, x_3. However, all choices of r prime to p give rise to isomorphic groups F/M_r. Assuming $p \neq 2, 3, 7$, we can choose r such that $r^3 \not\equiv \pm 1 \pmod{p}$. In particular, we may always take $r = 2$. Then M_1 and M_r are not conjugate under $\mathrm{Aut}(F)$.

In order to see this it suffices to show that no isomorphism of F/M_r onto F/M_1 lifts to an automorphism of F. One isomorphism of F/M_r onto F/M_1 is obtained by mapping x, y, z to x^r, y^r, z^r, respectively. This induces an automorphism of their common quotient $F/M_1M_r = F/F'F^p$, with determinant r^3 when the latter is viewed as a vector space over the field of p elements. Any other isomorphism of F/M_r onto F/M_1 is obtained by composing the one described with an automorphism of F/M_1. Since the latter has p-power order, and hence determinant one on $F/F'F^p$, we conclude that every isomorphism of F/M_r onto F/M_1 induces an automorphism of $F/F'F^p$ with determinant r^3. Because automorphisms of F induce maps of determinant ± 1 on F/F' viewed as a free $\mathbb{Z}$-module, and $r^3 \not\equiv \pm 1 \pmod{p}$, they cannot induce any isomorphism of F/M_r onto F/M_1.

A more careful analysis, such as that in the proof of the Theorem below, would reveal that for any odd prime p (thus including 3 and 7), the subgroups M_r and M_s are conjugate under $\mathrm{Aut}(F)$ if and only if $r \equiv \pm s \pmod{p}$. We leave the details to the interested reader and only suggest to use the fact that the subgroups $\langle G', x\rangle$ and $\langle G', x, y\rangle$ of $G = F/M_1$ are characteristic. In fact, according to [DH75], $\mathrm{Aut}(G)$ is generated by the automorphism determined by $x \mapsto x$, $y \mapsto xy$, $z \mapsto yz$ together with the p^9 central automorphisms, which induce the identity map on G/G'.

In the two-generated example which we present now the group of automorphisms of the finite quotients is not a p-group. Hence the proof is more involved, and we formally state the result as a theorem.

Theorem. *Let $F = \langle x, y\rangle$ be the free nilpotent group of rank two and class three, and let p be a prime greater than three. For $r = 1, \ldots, p-1$*

set

$$N_r = \langle x^{p^2}, y^p, x^{-rp}[y,x,x], [y,x,y]\rangle^F,$$

where the superscript F denotes taking the normal closure in F. Then F/N_r is a p-group of order p^4, class three and exponent p^2. All quotients F/N_r are isomorphic. However, N_r and N_s belong to the same orbit under $\mathrm{Aut}(F)$ *if and only if $r = s$ or $r = p - s$.*

Proof. It is well known that each element of F can be written as $x^i y^j [y,x]^k [y,x,x]^l [y,x,y]^m$ for uniquely determined integers i,j,k,l,m, see [Hal59, Theorem 11.2.4]. It is then easy to deduce that each coset of $K = \langle x^{p^2}, y^p, [y,x,y]\rangle^F$ in F has a unique representative of the form $x^i y^j [y,x]^k [y,x,x]^l$, with $0 \le i < p^2$ and $0 \le j,k,l < p$. This also follows from a general result giving $\mathbb{F}_p$-bases, in terms of basic commutators and their powers, for the factors of the lower p-central series of a free group, see [Sco91, Lemmas 1.11 and 1.12], for instance. In particular, K has index p^5 in F, and hence $N_r = \langle K, x^{-rp}[y,x,x]\rangle$ has index p^4 in F. Clearly, F/N_r has class three and exponent p^2.

We will determine all endomorphisms of F which map N_r into N_s and induce an isomorphism between the quotient groups F/N_r and F/N_s. Since M/N_r, where $M = \langle x^p, y\rangle^F$, is the only abelian maximal subgroup of F/N_r, we may restrict our attention to endomorphisms which map M into itself. Thus, let ψ be an endomorphism of F such that $\psi(x) = x^i y^j c$ and $\psi(y) = y^k d$, for some integers i,j,k and some $c,d \in F'F^p$. We may also assume that i and k are prime to p, because this is a necessary condition for inducing an isomorphism of F/N_r onto F/N_s and, in particular, an automorphism of $F/F'F^p$.

As a special case of [Hup67, Hilfssatz III.10.9(b)] or [LGM02, Corollary 1.1.7(i)], if a,b are elements of a p-group G of class less than p, and if the normal closure of b is abelian of exponent p, then $(ab)^p = a^p$. Since $\langle y, [y,x], [y,x,x], K\rangle/K$, the normal closure of yK in F/K, is abelian of exponent p, and because of standard commutator identities, we have

$$\begin{aligned}
&\psi(x^{p^2}) = ((x^i y^j c)^p)^p \equiv (x^{ip})^p \equiv 1 \pmod{K}\\
&\psi(y^p) = (y^k d)^p \equiv 1 \pmod{K}\\
&\psi([y,x,y]) = [y^k d, x^i y^j c, y^k d] \equiv [y,x,y]^{ik^2} \equiv 1 \pmod{K}.
\end{aligned}$$

Thus, ψ maps K into itself. Because of our assumption that i and k are prime to p, it induces an automorphism of $F/F'F^p$, and hence an automorphism of F/K, since the former is the Frattini quotient of the latter. Finally, we have

$$\begin{aligned}
\psi(x^{-rp}[y,x,x]) &= (x^i y^j c)^{-rp}[y^k d, x^i y^j c, x^i y^j c]\\
&\equiv x^{-irp}[y,x,x]^{i^2 k} \pmod{K}.
\end{aligned}$$

Consequently, ψ maps N_r into N_s if and only if $x^{-irp}[y,x,x]^{i^2k}$ equals a power of $x^{-sp}[y,x,x]$, that is, if and only if $iks \equiv r \pmod p$. If this condition is met, and it certainly can by a suitable choice of i and k, then ψ induces an isomorphism of F/N_r onto F/N_s, as desired.

It remains to see when the endomorphism ψ of F is an automorphism. Recall that F, being a finitely generated nilpotent group, is hopfian, that is, each surjective endomorphism of F is an automorphism [MKS76, Theorem 5.5]. Thus, ψ is an automorphism if and only if it is surjective, that is, if and only if it induces an automorphism of its Frattini quotient $F/\Phi(F) = F/F'$. It follows that ψ is an automorphism of F if and only if $ik = \pm 1$. Consequently, N_r and N_s belong to the same orbit under $\mathrm{Aut}(F)$ if and only if $r \equiv \pm s \pmod p$. □

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